Spectral Gap of Stable Commutator Length

Lvzhou Chen

Department of Mathematics
University of Chicago

AMS Sectional Meeting

The stable commutator length

Dictionary:

Group theory	G	γ	$\gamma \in [G, G]$
Topology	$X_G = K(G, 1)$	C_γ	C_γ bounds a surface

Definition (Geometric)

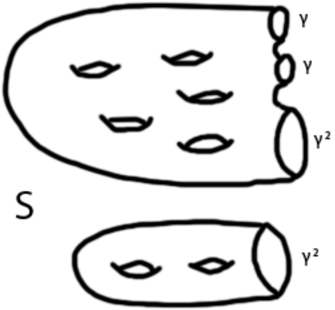
The stable commutator length

$$\mathrm{scl}_G(\gamma) := \inf_{S \text{ admissible}} \frac{-\chi(S)}{2 \cdot n(S)}, \quad \forall \gamma \in [G, G]$$

Example

Having an admissible surface like S exhibits

$$\mathrm{scl}(\gamma) \leq \frac{14}{2 \cdot 6} = \frac{7}{6}.$$



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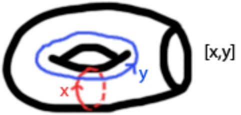
Example

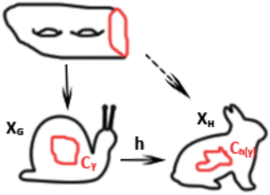
Figure shows $\mathrm{scl}([x,y]) \le \frac{1}{2}$.



Proposition

scl is monotone and characteristic: for $h: G \to H$,

$\mathrm{scl}_G(\gamma) \ge \mathrm{scl}_H(h(\gamma))$; *"$=$" if h is an isomorphism.*



Computing scl

Theorem

$\mathrm{scl}_G \equiv 0$ *if* G *is*

1. *(Burger–Monod'02) an "irreducible lattice of higher rank Lie groups",*
2. *(Johnson, Trauber, Gromov) or amenable.*

Theorem

scl_G *is (non-trivially) rational and computable if* G *are*

1. *(Calegari'08)* F_n $(n \geq 2)$, *or* $*G_i$ *with* G_i *abelian,*
2. *(Chen'16)* $*G_i$ *with* $\mathrm{scl}_{G_i} \equiv 0$,
3. *(Susse'13)or certain amalgams of abelian groups.*

scl spectrum of free groups (based on `scallop`)

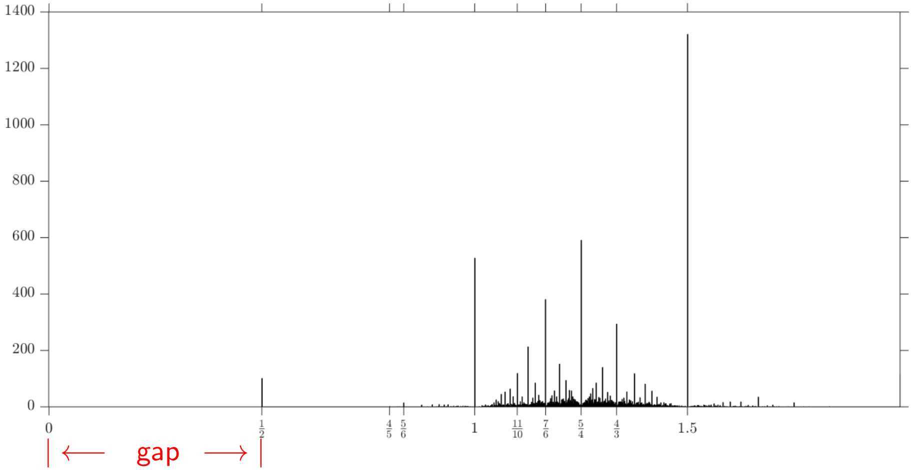


Figure: Values of scl on 10,000 random alternating words of length 36. (Cal[1])

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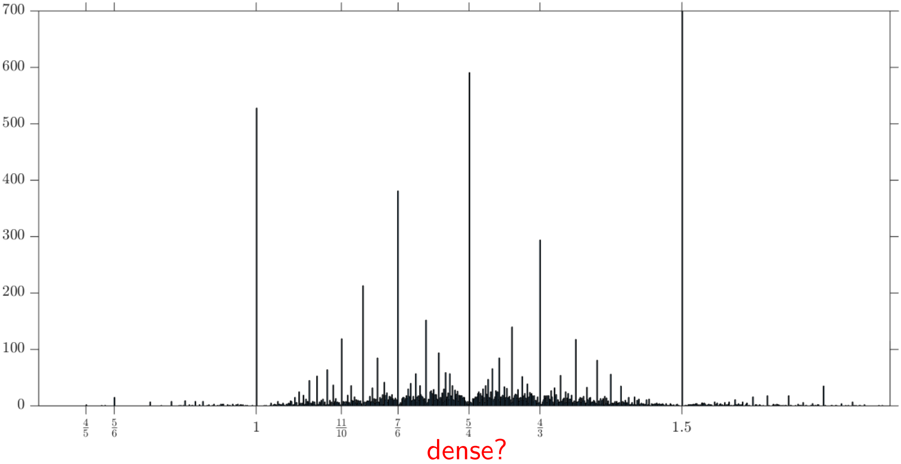

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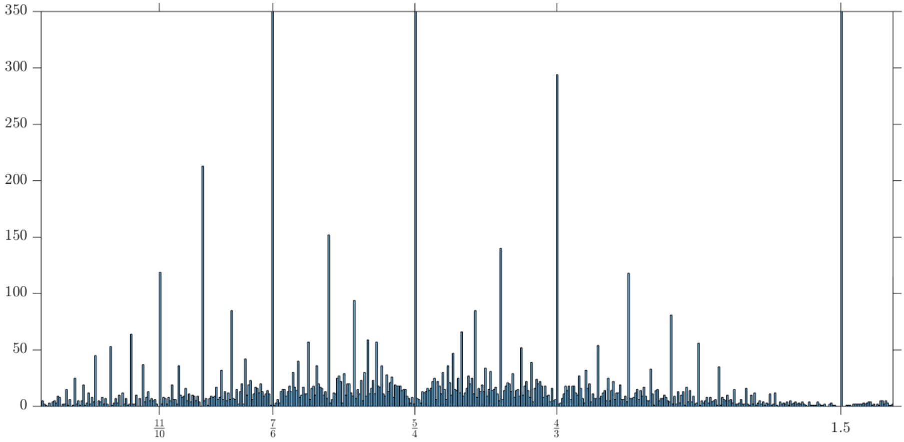

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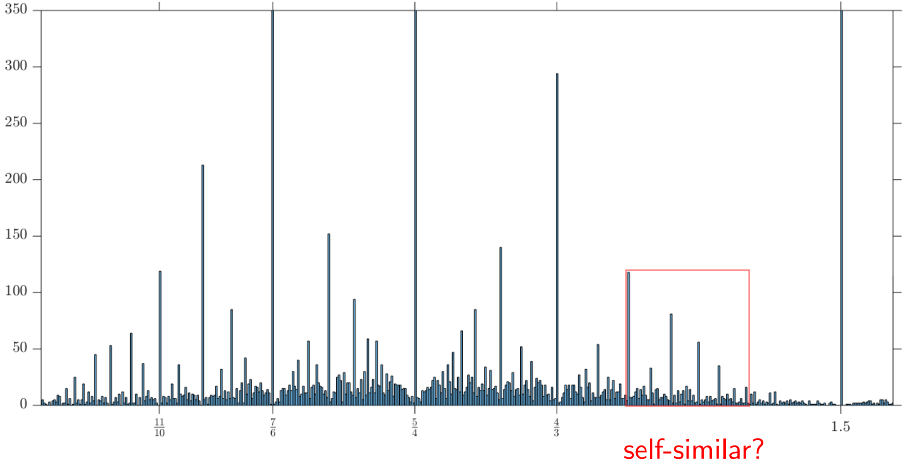

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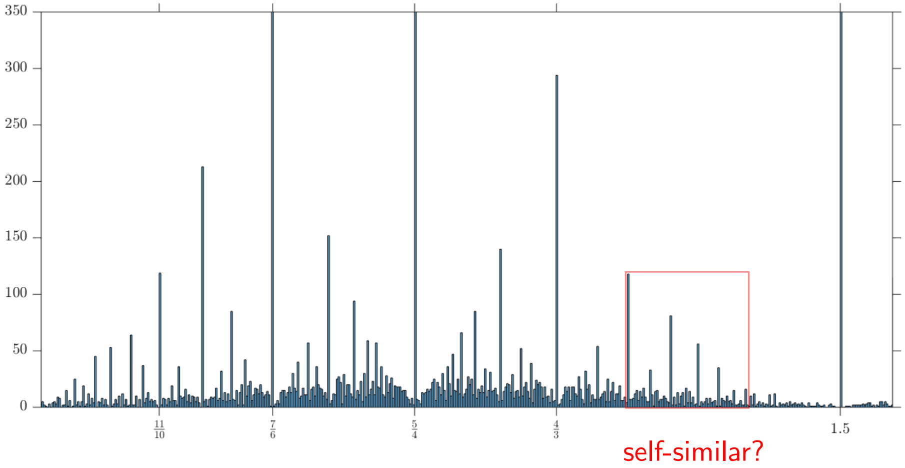

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Spectral gap theorems

Theorem (Duncan–Howie'91)

For $\gamma \neq id \in [F_k, F_k]$ ($k \geq 2$), $\mathrm{scl}(\gamma) \geq \frac{1}{2}$. Similar results for $G = *G_i$ with G_i locally indicable.

Theorem

1. *(Chen'16, Ivanov–Klyachko'17)Weaken locally indicable to torsion-free*
2. *(Chen'16) If torsion exists, we have lower bound $\frac{1}{2} - \frac{1}{n}$, where $n =$smallest torsion. Sharp for $\mathrm{scl}_{A*B}([a, b])$ with $a \in A$ and $b \in B$.*

Corollary

In F_k ($k \geq 2$), we have $\mathrm{scl}([x, y]) = \frac{1}{2}$ unless x, y commute.

Remark

Spectral gap exists for hyperbolic groups, MCGs, RAAGs, $BS(m, l)$, etc.

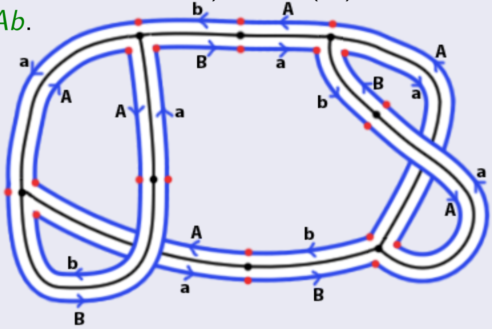
A new proof of spectral gap theorem

Theorem (Duncan–Howie)

For $\gamma \neq id \in [F_k, F_k]$ $(k \geq 2)$, $\mathrm{scl}(\gamma) \geq \frac{1}{2}$.

Proof (Chen).

Any admissible S, may assume (by Culler's theorem) $S = S(Y)$ for a fatgraph Y. E.g. $\gamma = aBaBabAAAb$.



Recall $\mathrm{scl}(\gamma) = \inf_S \dfrac{-\chi(S)}{2n(S)}$.

Goal: Show $-\chi(S) \geq n(S)$.

1. $-\chi(S) = -\chi(Y) = e - v$.
2. **Key**: $v \leq e - n$?

Proof (continued).

Key: $v \le e - n$?

Focus on $k = 2$ and $\gamma = aBaBabAAAb$. Then word length $L = 10$.

Label junctions in γ cyclically:

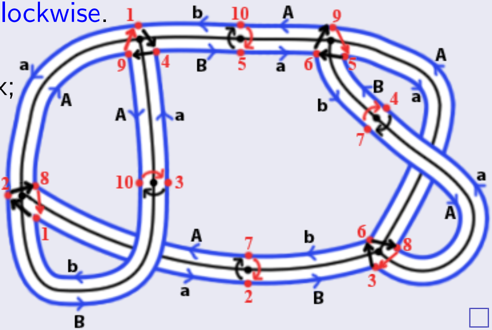
$\,_1a_2B_3a_4B_5a_6b_7A_8A_9A_{10}b$,

then pull back the labels to ∂S.

At each vertex, connect red dots clockwise.

Observe:

1. ≥ 1 descending at each vertex;
2. each edge contributes 1 descending, with n exceptions contributing 0 (companioned by $L \to 1$). $10 \to 1$



Thus $v \le \#\text{descendings} = e - n$. □

Open questions

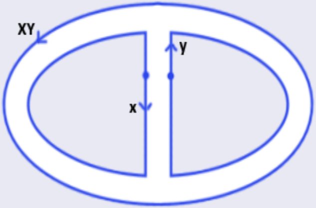
Conjecture (Calegari–Walker)

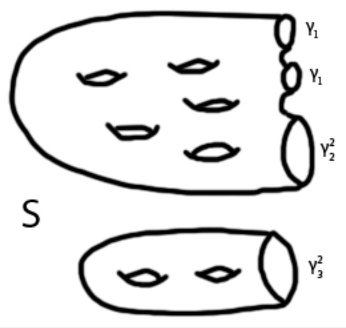
In F_k $(k \geq 2)$, for any $c = \gamma_1 + \ldots + \gamma_m$,

$$\mathrm{scl}(c) \geq \frac{1}{2}$$

unless c bounds annuli.

Special case: $\mathrm{scl}(x + y + XY) = \frac{1}{2}$ *unless x, y commute?*





Question

Is there a gap after $1/2$ in the spectrum?

Open questions

Question (Kervaire 1963)

G non-trivial. Is $\langle G, t | w\rangle$ non-trivial for any $w \in G * \langle t\rangle$?

Theorem (Klyachko 1993)

Yes for G torsion-free. Proof using "car motion", which has certain similarity to our proof of spectral gap theorem.

Question (Kervaire)

What about $\langle G, t_1, \ldots, t_n | w_1, \ldots, w_n\rangle$ for any $w_i \in G * \langle t_1, \ldots, t_n\rangle$? Related/Similar to Calegari–Walker conjecture?

For Further Reading

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scl.
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L. Chen
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R. Fenn and C. Rourke
Klyachko's methods and the solution of equations over torsion-free groups.
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A. Klyachko
A funny property of sphere and equations over groups.
Communications in algebra, **21**, no. 7 (1993): 2555–2575.